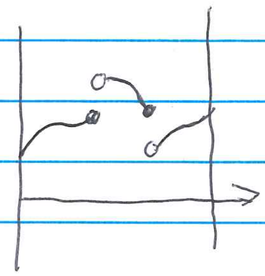
Sept 14, 2022

Week 2.

(Cont'd)

- Continuous functions are integrable on any $[a, b]$.
- piecewise continuous functions are integrable.



graph of a piecewise continuous function.

Double Integral

The theory of double integral is essentially the same as the single integral.

Let $R = [a, b] \times [c, d]$ be a rectangle and f a function on R.

A partition P on R is a collection of points satisfying

$$\{x_0, x_1, \ldots, x_n, y_0, y_1, \ldots y_m : a = x_0 < \ldots < x_n = b, \; c = y_0 < \ldots < y_m = d\}$$

P divides R into subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], \quad i = 1, \ldots, n, \; j = 1, \ldots m.$$

$$\|P\| = \max\{\Delta x_1, \ldots, \Delta x_n, \Delta y_1, \ldots, \Delta y_m\}$$

A tag is a collection of points $\{P_{ij}\}$, $P_{ij} \in R_{ij}$.

A Riemann sum of f with respect to P =

$$S(f, P) = \sum_{i,j} f(P_{ij}) |R_{ij}|, \qquad |R_{ij}| = \Delta x_i \Delta y_j.$$

A function f is integrable if there is a number I such that all Riemann sums come close to I when $\|P\|$ is small.

Notation: I usually is written as

$$\iint_R f, \quad \iint_R f(x,y)\, dA, \quad \iint_R f(x,y)\, dA(x,y).$$

When $f \geq 0$, $S(f, P)$ is the approximate volume of the solid bounded between the graph of f, and the xy-plane over R.

- unbounded functions are non-integrable,
- $$\varphi(x,y) = \begin{cases} 1 & x, y \text{ rationals} \\ 0 & \text{otherwise} \end{cases}$$

  is a bdd, non-integrable functions
- Continuous functions are integrable
- piecewise continuous functions are integrable.

  (those functions are continuous except jumping across some curves!)

Fubini's Theorem Let f be a continuous function on R. Then

$$\iint_R f(x,y)\,dA(x,y) = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy .$$

Fubini's theorem reduces double integral to 2 single integrals.

e.g. Evaluate $\iint_R xy^2\,dA$, $R = [0,2] \times [0,1]$.

By Fubini's thm,

$$\iint_R xy^2\,dA = \int_0^2 \left(\int_0^1 xy^2\,dy \right) dx$$

$$= \int_0^2 x \frac{y^3}{3}\Big|_0^1 dx$$

$$= \int_0^2 \frac{x}{3}\,dx$$

$$= \frac{x^2}{6}\Big|_0^2$$

$$= \frac{2}{3} .$$

Alternatively, we could do it in dx first.

$$\iint_R xy^2\, dA = \int_0^1 \left(\int_0^2 xy^2\, dx \right) dy$$

$$= \int_0^1 \frac{x^2}{2} y^2 \Big|_0^2 \, dy$$

$$= \int_0^1 2y^2\, dy$$

$$= \frac{2}{3} y^3 \Big|_0^1$$

$$= \frac{2}{3} \quad \#$$

e.g. Evaluate $\displaystyle\iint_{[0,1]\times[0,\pi]} x \sin xy \, dA$

$$\iint_{[0,1]\times[0,\pi]} x \sin xy \, dA = \int_0^{\pi} \left(\int_0^1 x \sin xy \, dx \right) dy$$

As

$$\int_0^1 x \sin xy \, dx = \int_0^1 x \left(-\frac{\cos xy}{y} \right)' dx$$

$$= x \frac{-\cos xy}{y} \Big|_0^1 - \int_0^1 \frac{-\cos xy}{y} \cdot dx$$

$$= -\frac{\cos y}{y} + \int_0^1 \frac{\cos xy}{y} \, dx$$

$$= -\frac{\cos y}{y} + \frac{\sin y}{y^2},$$

$$\iint_{[0,1]\times[0,\pi]} x \sin xy \, dA = \int_0^{\pi} \left(\frac{-\cos y}{y} + \frac{\sin y}{y^2} \right) dy ,$$

but then we get stuck. So we switch order.

$$\iint_{[0,1]\times[0,\pi]} x \sin xy \, dA = \int_0^1 \left(\int_0^{\pi} x \sin xy \, dy \right) dx$$

$$= \int_0^1 x \left. \frac{-\cos xy}{x} \right|_0^{\pi} dx$$

$$= \int_0^1 (-\cos \pi x + 1) \, dx$$

$$= \left. \left(-\frac{\sin \pi x}{\pi} + x \right) \right|_0^1$$

$$= 1 \quad \#$$

e.g. Find the volume of the solid bounded by $z = 10 + x^2 + 3y^2$ and the xy-plane over $R: 0 \le x \le 1, \ 0 \le y \le 2$.

The volume is

$$\iint_{[0,1]\times[0,2]} (10 + x^2 + 3y^2) \, dA = \int_0^1 \int_0^2 (10 + x^2 + 3y^2) \, dy \, dx$$

$$= \int_0^1 \left. (10y + x^2 y + y^3) \right|_0^2 dx$$

$$= \int_0^1 (20 + 2x^2 + 8) \, dx$$

$$= \left. \left(28x + \frac{2}{3}x^3 \right) \right|_0^1$$

$$= 86/3 \quad \#$$